Denotational Semantics

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Lectures for Part II CST 2026/2027

Practicalities

- My mail: im496@cam.ac.uk.
- Do not hesitate to ask questions!
- Feel free to give me feedback at any point!

Introduction

What is this course about?

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Goal: give an abstract and compositional (mathematical) model of programs.

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- Documentation: precise but intuitive, machine-independent specification.
- Language design: feedback from semantics (functional programming, monads & handlers, linearity…).
- Rigour: powerful way to justify formal methods.

Styles of formal semantics

- Operational
- Axiomatic
- Denotational

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- **Axiomatic**: meaning of a program in terms of a *program logic* to reason about it (see Part II Hoare Logic & Model Checking).
- **Denotational**: meaning of a program defined abstractly as object of some suitable *mathematical structure* (see this course).

Denotational semantics in a nutshell

	$\overset{[\![-]\!]}{\longrightarrow}$	
Syntax		Semantics
Program P	$\mapsto$	Denotation $[\![P]\!]$
Arithmetic expression	$\mapsto$	Number
Boolean circuit	$\mapsto$	Boolean function
Recursive program	$\mapsto$	Partial recursive function
	...	

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Type	$\mapsto$	Domain
Program	$\mapsto$	Continuous functions between domains

Properties of denotational semantics

Abstraction

- mathematical object, implementation/machine independent;
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Compositionality

- The denotation of a whole is defined using the *denotations* of its parts;
- $[\![P]\!]$ represents the contribution of P to *any* program containing P;
- More flexible and expressive than whole-program semantics.

Introduction

A basic example

Programs

$$C \in \mathbf{Prog} ::= \texttt{skip} \mid L := A \mid C; C \mid \texttt{if } B \texttt{ then } C \texttt{ else } C \mid \texttt{while } B \texttt{ do } C$$

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Arithmetic expressions

$$A \in \mathbf{Aexp} ::= \underline{n} \mid L \mid A + A \mid \ldots$$

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ranges over *integers*

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Arithmetic expressions

$$A \in \mathbf{Aexp} ::= \underline{n} \mid L \mid A + A \mid \ldots$$

Boolean expressions

$$B \in \mathbf{Bexp} ::= \texttt{true} \mid \texttt{false} \mid A = A \mid \neg B \mid \ldots$$

Programs

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Denotation functions – naïvely

$$\mathcal{A} : \quad \mathbf{Aexp} \to \mathbb{Z}$$

where

$$\mathbb{Z} \quad = \quad \{\ldots, -1, 0, 1, \ldots\}$$

Denotation functions – naïvely

$$\begin{aligned} \mathcal{A} : &\quad \mathbf{Aexp} \to \mathbb{Z} \\ \mathcal{B} : &\quad \mathbf{Bexp} \to \mathbb{B} \end{aligned}$$

where

$$\begin{aligned} \mathbb{Z} &= \{\ldots, -1, 0, 1, \ldots\} \\ \mathbb{B} &= \{\mathrm{true}, \mathrm{false}\} \end{aligned}$$

Arithmetic expressions?

$$\begin{aligned} \mathcal{A}[\![\underline{n}]\!] &= n \\ \mathcal{A}[\![A_1 + A_2]\!] &= \mathcal{A}[\![A_1]\!] + \mathcal{A}[\![A_2]\!] \end{aligned}$$

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\mathcal{A}[\![L]\!] &= ???
\end{aligned}
$$

Denotation functions – less naïvely

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Denotation functions – less naïvely

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$$\mathcal{A} : \textbf{Aexp} \to (\text{State} \to \mathbb{Z})$$
$$\mathcal{B} : \textbf{Bexp} \to (\text{State} \to \mathbb{B})$$
$$\mathcal{C} : \textbf{Prog} \to (\text{State} \to \text{State})$$

where

$$\mathbb{Z} = \{\ldots, -1, 0, 1, \ldots\}$$
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Semantics of arithmetic expressions

$$\mathcal{A}[\![\underline{n}]\!] = \lambda s \in \text{State} \,.\, n$$

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\mathcal{A}[\![L]\!] &= \lambda s \in \text{State}.\, s(L)
\end{aligned}
$$

Semantics of boolean expressions

$$\mathcal{B}[\![\texttt{true}]\!] = \lambda s \in \text{State}.\ \text{true}$$

$$\mathcal{B}[\![\texttt{false}]\!] = \lambda s \in \text{State}.\ \text{false}$$

$$\mathcal{B}[\![A_1 = A_2]\!] = \lambda s \in \text{State}.\ \text{eq}\,(\mathcal{A}[\![A_1]\!](s), \mathcal{A}[\![A_2]\!](s))$$

$$\text{where } \text{eq}\,(a, a') = \begin{cases} \text{true} & \text{if } a = a' \\ \text{false} & \text{if } a \neq a' \end{cases}$$

$$\mathcal{C}[\![\texttt{skip}]\!] \quad = \quad \lambda s \in \mathrm{State}.\ s$$

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$$\text{where } \text{if}\,(b, x, x') = \begin{cases} x & \text{if } b = \text{true} \\ x' & \text{if } b = \text{false} \end{cases}$$

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This is compositionality!

where $\text{if}\,(b, x, x') = \begin{cases} x & \text{if } b = \text{true} \\ x' & \text{if } b = \text{false} \end{cases}$

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$$\mathcal{C}[\![C; C']\!] = \mathcal{C}[\![C']\!] \circ \mathcal{C}[\![C]\!] = \lambda s \in \text{State}.\ \mathcal{C}[\![C']\!]\,(\mathcal{C}[\![C]\!](s))$$

Introduction

A semantics for loops

Semantics of loops?

This is all very nice, but...

$$[\![\texttt{while}\ B\ \texttt{do}\ C]\!] = ???$$

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Remember:

- $\langle \texttt{while}\ B\ \texttt{do}\ C, s\rangle \rightsquigarrow \langle \texttt{if}\ B\ \texttt{then}\ (C; \texttt{while}\ B\ \texttt{do}\ C)\ \texttt{else}\ \texttt{skip}, s\rangle$
- we want compositional semantics: $[\![\texttt{while}\ B\ \texttt{do}\ C]\!]$ in terms of $[\![C]\!]$ and $[\![B]\!]$
- we want denotational semantics compatible with the operational semantics

Loop as a fixpoint

$$
\begin{aligned}
[\![\texttt{while}\ B\ \texttt{do}\ C]\!] &= [\![\texttt{if}\ B\ \texttt{then}\ (C; \texttt{while}\ B\ \texttt{do}\ C)\ \texttt{else}\ \texttt{skip}]\!] \\
&= \lambda s \in \mathrm{State}.\ \mathrm{if}\ ([\![B]\!](s), ([\![\texttt{while}\ B\ \texttt{do}\ C]\!] \circ [\![C]\!])(s), s)
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Loop as a fixpoint

$$\begin{aligned}[\![\texttt{while } B \texttt{ do } C]\!] &= [\![\texttt{if } B \texttt{ then } (C; \texttt{while } B \texttt{ do } C) \texttt{ else skip}]\!] \\ &= \lambda s \in \text{State}.\ \text{if } ([\![B]\!](s), ([\![\texttt{while } B \texttt{ do } C]\!] \circ [\![C]\!])(s), s)\end{aligned}$$

We don't have a direct definition for $[\![\texttt{while } B \texttt{ do } C]\!]$, but a fixed point equation!

$$[\![\texttt{while } B \texttt{ do } C]\!] = F_{[\![B]\!],[\![C]\!]}([\![\texttt{while } B \texttt{ do } C]\!])$$

where

$$\begin{aligned} F_{b,c} :\ (\text{State} \to \text{State}) &\to (\text{State} \to \text{State}) \\ w &\mapsto \lambda s \in \text{State}.\ \text{if } (b(s), (w \circ c)(s), s)\end{aligned}$$

Now we have a goal

- Why/when does $w = F_{b,c}(w)$ have a solution?
- What if it has several solutions? Which one should be our $[\![\texttt{while}\ B\ \texttt{do}\ C]\!]$?

Introduction

A taste of domain theory

Total functions are not enough

Forget about $\mathbf{State}$ for a second, consider these equations ($f \in \mathbb{Z} \to \mathbb{Z}$):

$$f(x) = f(x) + 1 \tag{1}$$
$$f(x) = f(x) \tag{2}$$

What about their fixed points?

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What about their fixed points?

- No function satisfies equation (1)!
- All functions satisfy equation (2)!

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has a unique solution: the function $\bot$ that is everywhere undefined

But

$$f(x) = f(x)$$

has even more solutions now - all partial functions. Which one should we pick?

‘Information order’ on partial functions

Partial order on $\mathbb{Z} \rightharpoonup \mathbb{Z}$:

$w \sqsubseteq w'$ iff for all $s \in \mathbb{Z}$, if w is defined at s, so is w' and moreover $w(s) = w'(s)$
iff the graph of w is included in the graph of w'

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$\bot$ is the least solution to $f(x) = f(x)$ making it a ‘canonical’ choice.

Back to loops (An example)

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should be some w such that:

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should be some w such that:

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That is, we are looking for a fixed point of the following function F:

$$\begin{array}{rcl} F : (\text{State} \rightharpoonup \text{State}) & \to & (\text{State} \rightharpoonup \text{State}) \\ w & \mapsto & \lambda[X \mapsto x, Y \mapsto y]. \begin{cases} [X \mapsto x, Y \mapsto y] & \text{if } x \leq 0 \\ w\,([X \mapsto x - 1, Y \mapsto x \cdot y]) & \text{if } x > 0 \end{cases} \end{array}$$

Approximating the least fixed point

$$F(w) = \lambda[X \mapsto x, Y \mapsto y].\begin{cases}[X \mapsto x, Y \mapsto y] & \text{if } x \leq 0 \\ w\,([X \mapsto x - 1, Y \mapsto x \cdot y]) & \text{if } x > 0\end{cases}$$

Define recursively $w_n = F^n(w)$, that is $w_0 = \bot$ and $w_{n+1} = F(w_n)$.

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$$w_1[X \mapsto x, Y \mapsto y] = \begin{cases}[X \mapsto x, Y \mapsto y] & \text{if } x \leq 0 \\ \text{undefined} & \text{if } x \geq 1\end{cases}$$

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Define recursively $w_n = F^n(w)$, that is $w_0 = \bot$ and $w_{n+1} = F(w_n)$.

$$w_2[X \mapsto x, Y \mapsto y] = \begin{cases}[X \mapsto x, Y \mapsto y] & \text{if } x \leq 0 \\ [X \mapsto 0, Y \mapsto y] & \text{if } x = 1 \\ \text{undefined} & \text{if } x \geq 2\end{cases}$$

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$$w_3[X \mapsto x, Y \mapsto y] = \begin{cases}[X \mapsto x, Y \mapsto y] & \text{if } x \leq 0 \\ [X \mapsto 0, Y \mapsto y] & \text{if } x = 1 \\ [X \mapsto 0, Y \mapsto 2y] & \text{if } x = 2 \\ \text{undefined} & \text{if } x \geq 3\end{cases}$$

Approximating the least fixed point

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$$w_n[X \mapsto x, Y \mapsto y] = \begin{cases}[X \mapsto x, Y \mapsto y] & \text{if } x \leq 0 \\ [X \mapsto 0, Y \mapsto (x!) \cdot y] & \text{if } 0 < x < n \\ \text{undefined} & \text{if } x \geq n\end{cases}$$

$$w_0 \sqsubseteq w_1 \sqsubseteq \ldots \sqsubseteq w_n \sqsubseteq \ldots$$

Approximating the least fixed point

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$$w_0 \sqsubseteq w_1 \sqsubseteq \ldots \sqsubseteq w_n \sqsubseteq \ldots \sqsubseteq w_\infty?$$

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$$w_0 \sqsubseteq w_1 \sqsubseteq \ldots \sqsubseteq w_n \sqsubseteq \ldots \sqsubseteq w_\infty$$

$$w_\infty[X \mapsto x, Y \mapsto y] = \bigsqcup_{n \in \mathbb{N}} w_n = \begin{cases}[X \mapsto x, Y \mapsto y] & \text{if } x \leq 0 \\ [X \mapsto 0, Y \mapsto (x!) \cdot y] & \text{if } x > 0\end{cases}$$

We have our semantics

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$$= \begin{cases} [X \mapsto x, Y \mapsto y] & \text{if } x \leq 0 \\ [X \mapsto 0, Y \mapsto 1 \cdot y] & \text{if } x = 1 \\ [X \mapsto 0, Y \mapsto (x-1)! \cdot x \cdot y] & \text{if } x > 0 \end{cases} \quad \text{(definition of } w_\infty\text{)}$$

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We have our semantics

$$F(w_\infty)[X \mapsto x, Y \mapsto y] = \begin{cases} [X \mapsto x, Y \mapsto y] & \text{if } x \leq 0 \\ w_\infty[X \mapsto x - 1, Y \mapsto x \cdot y] & \text{if } x > 0 \end{cases} \quad \text{(definition of } F\text{)}$$

$$= \begin{cases} [X \mapsto x, Y \mapsto y] & \text{if } x \leq 0 \\ [X \mapsto 0, Y \mapsto 1 \cdot y] & \text{if } x = 1 \\ [X \mapsto 0, Y \mapsto (x-1)! \cdot x \cdot y] & \text{if } x > 0 \end{cases} \quad \text{(definition of } w_\infty\text{)}$$

$$= w_\infty[X \mapsto x, Y \mapsto y]$$

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$$= w_\infty[X \mapsto x, Y \mapsto y]$$

- $F(w_\infty) = w_\infty$ *i.e.* w_∞ is a fixed point of F;
- It is the least fixed point;
- Using w_∞ as the denotation of while is compatible with the operational semantics!

$$[\![\texttt{while } X > 0 \texttt{ do } (Y := X * Y; X := X - 1)]\!] = w_\infty$$

The rest of this course

The course can be roughly divided into two parts:

I: domain theory

II: denotational semantics for the language PCF